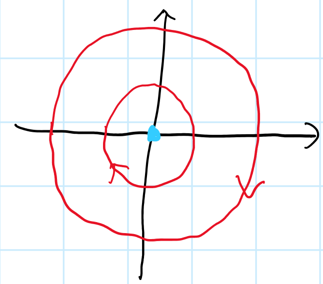
5.7 periodic solutions and Poincare-Bendixson

Wednesday, March 24, 2021 12:07 PM

Recall: A 2D linear autonomous system $\dot{X}(t) = AX(t)$, where $A \in \mathbb{R}^{2\times 2}$ and the eigenvalues of A are purely imaginary, has a centre at 0 and periodic solutions around it.

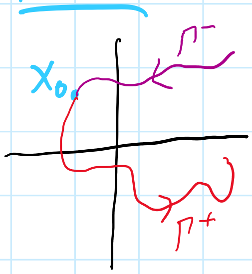
e.g. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda_{1,2} = \pm i$



When does a nonlinear system have periodic solutions?

Notation: Consider the 2D autonomous system



$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y),$$

where f and g have continuous partial derivatives in some region of the plane.

The solution trajectory $\Gamma(X_0, t)$ is a solution to the system where $X_0 = (x_0, y_0) = (x(t_0), y(t_0))$.

$\Gamma^+(X_0, t)$ is the positive orbit, the part of $\Gamma(X_0, t)$ where $t \geq t_0$

$\Gamma^-(X_0, t)$ is the negative orbit, the part of $\Gamma(X_0, t)$, where $t \leq t_0$

If solutions are bounded, then they approach limiting sets a $t \to -\infty$ or $t \to \infty$

The α-limit set $\alpha(X_0)$ is the set of points approached by Γ^- as $t \to -\infty$, i.e. $(x_e, y_e) \in \alpha(X_0)$ iff $\exists$ a decreasing sequence of times $\{t_i\}_{i=1}^{\infty}$, $t_i \to -\infty$

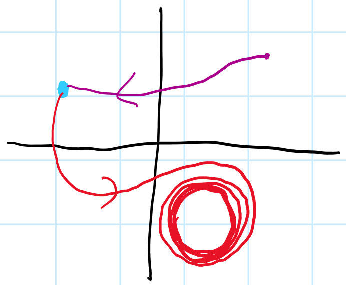
s.t. $\lim_{i\to\infty} (x(t_i), y(t_i)) = (x_e, y_e)$.

The ω-limit set $\omega(X_0)$ is the set of points approached by Γ^+ as $t \to +\infty$, i.e. $(x_e, y_e) \in \omega(X_0)$ iff $\exists$ an increasing sequence of times $\{t_i\}_{i=1}^{\infty}$, $t_i \to +\infty$

s.t. $\lim_{i\to\infty} (x(t_i), y(t_i)) = (x_e, y_e)$

i.e. $(x_e, y_e) \in \omega(X_0)$ iff $\exists$ an increasing sequence of times $\{t_i\}_{i=1}^{\infty}$, $t_i \to +\infty$

s.t. $\lim_{i \to \infty} (x(t_i), y(t_i)) = (x_e, y_e)$.



How do we classify and understand what these limiting sets look like?

Thm 5.6 (Poincaré-Bendixson)

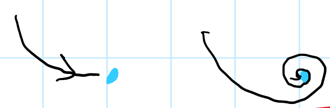
Let $\Gamma^+(X_0, t)$ be a positive orbit that remains in a closed and bounded region of the plane. Suppose $\omega(X_0)$ does not contain any equilibria. Then $\omega(X_0)$ is a periodic orbit.

Note: Some $\Gamma^+(X_0, t) = \omega(X_0)$, in which case $\Gamma^+(X_0, t)$ is periodic.

Thm 5.7 (Poincaré-Bendixson trichotomy)

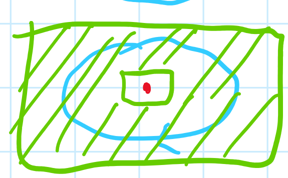
Let $\Gamma^+(X_0, t)$ be a positive orbit that remains in a closed and bounded region B of the plane. Suppose B contains a finite number of equilibria. Then either

(i) $\omega(X_0)$ is an equilibrium.



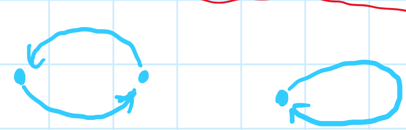
(ii) $\omega(X_0)$ is a periodic solution

Aside: periodic orbits must enclose at least 1 equilibrium.



$\to$ closed and bounded region

(iii) $\omega(X_0)$ contains a finite number of equilibria and a set Γ_i of trajectories whose α- and ω-limit sets consist of one of these equilibria $\forall\, \Gamma_i$.



Thm 5.8 (Bendixson's criteria)

Suppose $D \subset \mathbb{R}^2$ is open and simply connected. (no holes)

If $\operatorname{div}(f, g) \equiv \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically 0, and does not change sign in D,

Suppose $D \subset \mathbb{R}^2$ is open and simply connected.
If $\operatorname{div}(f,g) \equiv \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically 0, and does not change sign in D, then there are no periodic orbits of $\frac{dx}{dt} = f(x,y)$ $\frac{dy}{dt} = g(x,y)$ in D.

Thm 5.9 (Dulac's criterion)

Suppose $D \subset \mathbb{R}^2$ is open and simply connected, and $\underbrace{B}_{\text{Dulac function}}(x,y)$ has continuous partial derivatives in D.

If $\operatorname{div}(Bf, Bg) = \frac{\partial (Bf)}{\partial x} + \frac{\partial (Bg)}{\partial y}$ is not identically 0, and does not change sign in D,

then there are no periodic orbits of the system $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ in D.

Note: If we let $B(x,y) = 1$, then Dulac's criterion is Bendixson's criterion.

Aside: There is no general technique for finding a Dulac function
(viz integrating factor)

Ex 7.3.2 [Strogatz, Nonlinear Dynamics and Chaos, 2nd Ed]

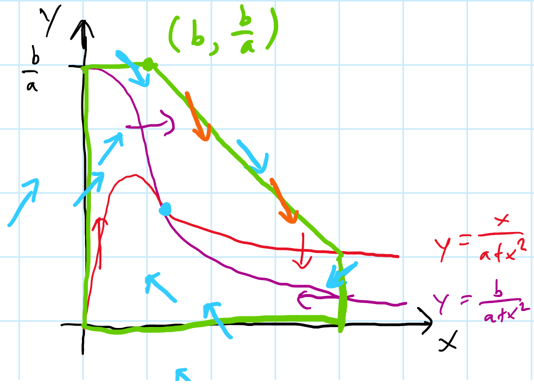
Cells use glycolysis to obtain energy by breaking by sugar.
Two of the intermediates are ADP (adenosine diphosphate) and F6P (fructose-6-phosphate)
A simplified model of the interactions is

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y, \quad a, b > 0$$

x-nullcline: $\dot{x} = 0 = -x + ay + x^2 y \Rightarrow y = \frac{x}{a + x^2}$

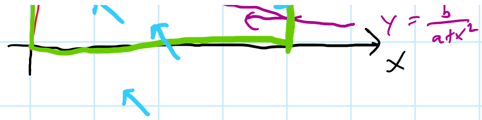
y-nullcline: $\dot{y} = 0 = b - ay - x^2 y \Rightarrow y = \frac{b}{a + x^2}$

Equilibrium at $\bar{x} = b$, $\bar{y} = \frac{b}{a + b^2}$



Note: The nullclines divide up the plane into regions with positive or negative $\dot{x}$ and $\dot{y}$

Note that $\dot{x} + \dot{y} = -x + ay + x^2 y + b - ay - x^2 y$
$= b - x$



Note that $\dot{x} + \dot{y} = -x + ay + x^2y + b - ay - x^2y$
$= b - x$

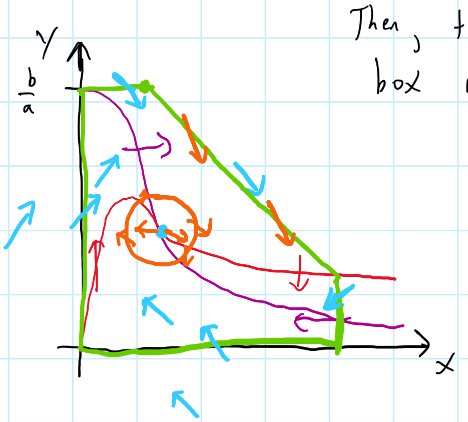
So if $x > b$, then $\dot{x} + \dot{y} < 0$

$\boxed{\dot{x} < -\dot{y}}$

Thus, all solutions starting in the green box are trapped within the green box

Suppose the equilibrium is an unstable node or spiral. Then we can cut out a little open disc around it, and solutions are still bounded.

Then, there must be a periodic solution in the green box minus the orange circle.



Note: This excision only works if the equilibrium is an unstable node or spiral. Does not work if it is stable or a saddle pt.

Ex. 5.17 Consider the following predator-prey model where both logistically grow in the absence of the other.

$$\frac{dx}{dt} = x(1 - ax - by)$$

$$\frac{dy}{dt} = y(1 + cx - dy), \quad a, b, c, d > 0.$$

Let $B(x,y) = \frac{1}{xy}$, which is continuously differentiable in $D = \{(x,y) \mid x > 0, y > 0\}$.

$$\operatorname{div}\left(Bx(1-ax-by),\ By(1+cx-dy)\right) = \operatorname{div}\left(\frac{1}{y}(1-ax-by),\ \frac{1}{x}(1+cx-dy)\right)$$

$$= \operatorname{div}\left(\frac{1}{y} - a\cdot\frac{x}{y} - b,\ \frac{1}{x} + c - d\cdot\frac{y}{x}\right)$$

$$= \frac{\partial\left[\frac{1}{y} - a\cdot\frac{x}{y} - b\right]}{\partial x} + \frac{\partial\left[\frac{1}{x} + c - d\cdot\frac{y}{x}\right]}{\partial y} = -\frac{a}{y} - \frac{d}{x} < 0 \text{ in } D.$$

$\Rightarrow$ By Dulac's criterion, there cannot exist any periodic solutions in D.